On Turbulence and Complex Dynamic in a Four-dimensional Peano-Hilbert Space

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ABSTRACT: *The work outlines a fairly model-independent scenario for a dynamic system to possess strange nonchaotic behaviour in the presence of quasi-periodic forcing. Implications for ergodicity and turbulence are discussed in connection with Sierpenski and peano-Hilbert spaces.*

I. Introduction

Recent investigations (**1–4**) revealed the existence of strange, yet nonchaotic attractors in quasi-periodically driven oscillators. These are fractal attractors which look topologically strange, yet possess negative Liapunov exponents. Also, recently Roessler *et al.* (**5**, **6**) gave reasons for anticipating new, strange chaotic phenomena in four dimensions. Now the simplest quasi-periodically forced oscillators require a four-dimensional phase space similar to the Ruelle, Takens and Newhouse chaos scenario (**7**). Hence it is of interest to investigate possible cross connections between all these different lines of thought.

In what follows, we outline a horseshoe-like discrete map (**7**) which may serve as a paradigm for strange, nonchaotic behaviour. We show that contracting, stretching and twist-folding the phase space leads to a distinct form of dynamics. The invariant set of this dynamics are Cantor-like objects and may be shown to represent a peano-Hilbert locally connected space (**8**). The immediate consequence of this picture is that we may anticipate a Poincaré map of a system with strange nonchaotic behaviour to have a fractal dimension, tending towards $d_c = 2$. In addition, we investigate the possibility of a complex dynamic in a Sierpenski space and its connection to strange nonchaotic, turbulent and ergodic behaviour.

II. A Periodically Twisted Horseshoe-like Map for Phase Space Deformation

Consider first the fractal construction of Fig. 1(a). The governing rule is to delete the middle cross of a unit square, then from the remaining four squares remove again the middle cross, and so on *ad infinitum*. It is easy to show that the Lebesque measure of the resulting set tends towards zero, while its Hausdorff capacity dimension tends towards $d_c = \log 4/\log 3 = 2(0.6309) \cong 1.261$. This is clearly the Cartesian product of two middle third Cantor sets (**7–11**). Now, the four corner squares

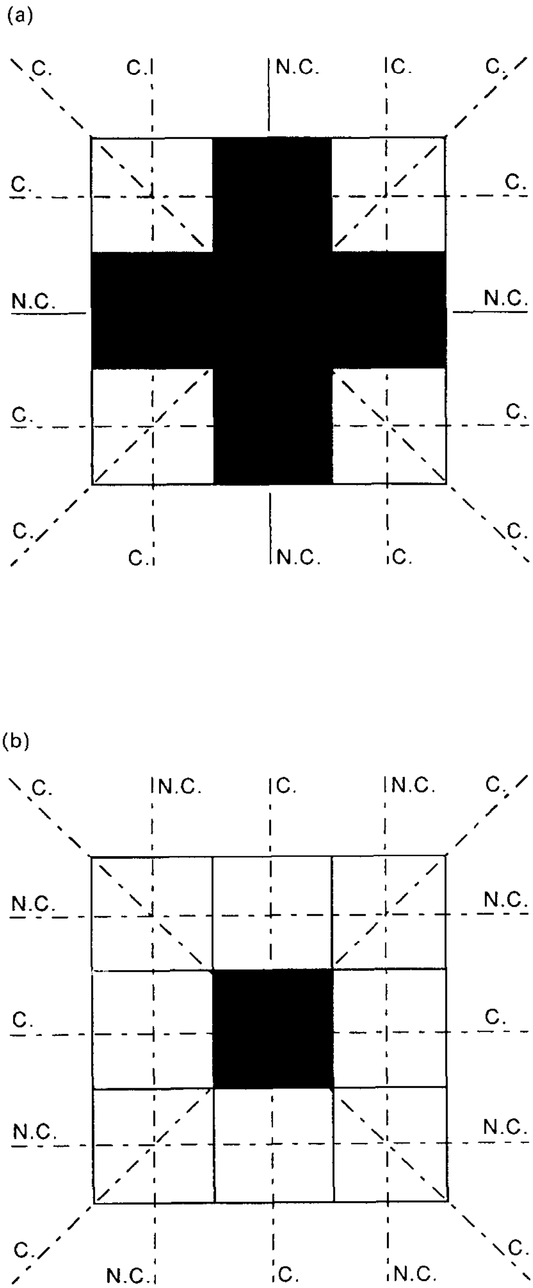
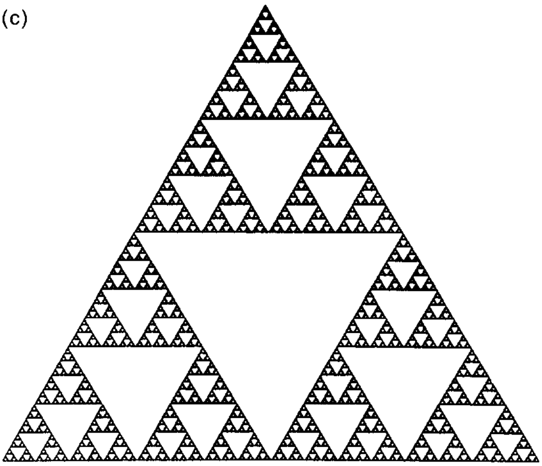


FIG. 1. (a, b) Two square fractal lattices with different symmetries. The first (a) has six directions of single triadic Cantor set. The second (b) has only four directions of single triadic Cantor set. (Note that C–C means triadic Cantor set direction while N.C–N.C means connected or empty space). (c) The Sierpenski gasket is the two-dimensional generalization of a one-dimensional triadic Cantor set. "Lefting" to higher dimensions leads to Table I.

(c)



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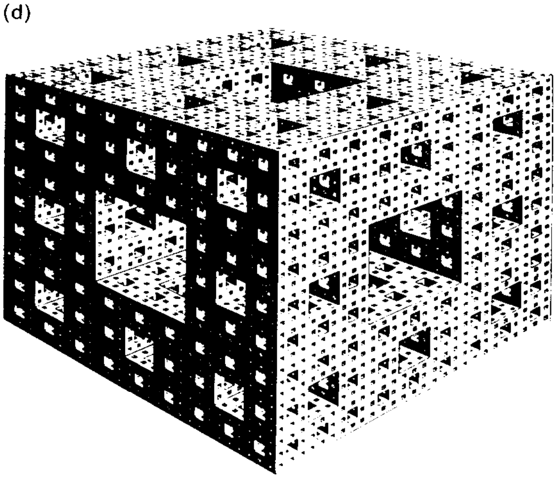


It is important to note that the Sierpenski gasket is invariant under a set of transformation and may be called quasi-symmetric under the action of a "quasi-group" (**23**). (d) The Menger sponge, named after K. Menger, the founder of statistical geometry. It is quite possible that micro space-time is a very similar structure (**22**, **24**). Note also that the generalization of Mandelbrot's foam, $d = 2.9656$, to four dimensions, leads to $d = 3.98869 \approx 4$.

of Fig. 1(a), which ultimately give rise to this Cantor set, correspond to the invariant set of a two-dimensional horseshoe map and are analogous to the intersection areas of the horseshoe and its pre-image (7). In fact, if the dissipation parameter of a one-dimensional Smale horseshoe is made equal to $\alpha = 7.05595$, then the fractal dimension will equal that of our two-dimensional map of Fig. 1(a). A dynamic which can be attached to this simple fractal construction is: pressing, stretching and bending in a C shape; then repeating the procedure *ad infinitum*. Suppose now that in addition to these actions we introduce a periodic torsional (twist) deformation before bending into an S shape. It is not difficult to see that an idealized form of the projection of such a torsional deformation of a long stretched rectangular strip would be similar to the nine elements of the 45° rotated square drawn on the original square of Fig. 2. This may also be demonstrated by folding a long twisted paper strip, as in Fig. 3. It is easily established that the capacity of the resulting geometrical set, as we repeat this specific deformation mechanism, is $d_c = \log 36/\log 6 = 2$. In what follows, we will attempt to make it plausible that the resulting Cantor set-like objects generated by this iteration represents a dynamic on a peano-Hilbert space (7), and are related to quasi-periodically forced two-

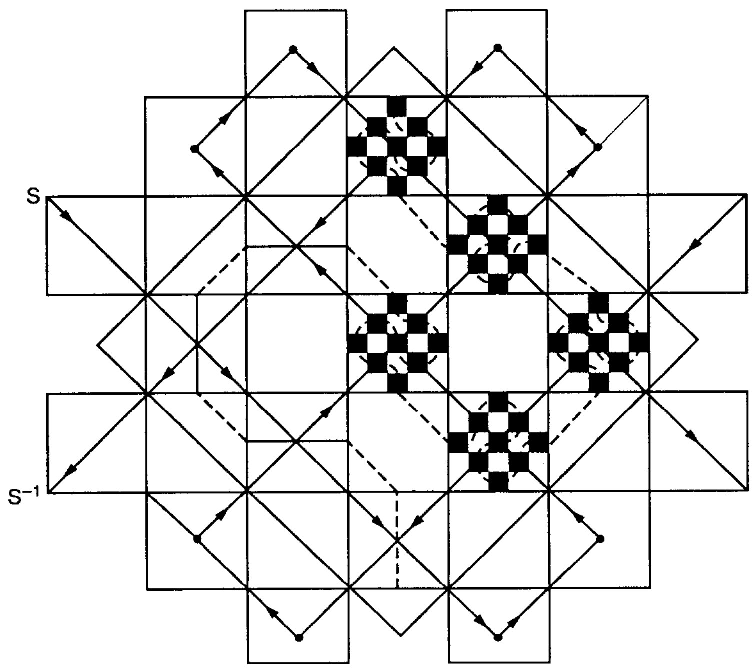


Fig. 2. A geometrical construction to explain the peano dynamics iteration. Two squares can be seen. These are the five by five large square ($d_c = \log 13/\log 5$) and the 45°-rotated square ($d_c = \log 36/\log 6$). In the first iteration only the squares at the intersections of S and S^{-1} are retained. In the second iteration only the nine black squares are retained and so on, *ad infinitum*.

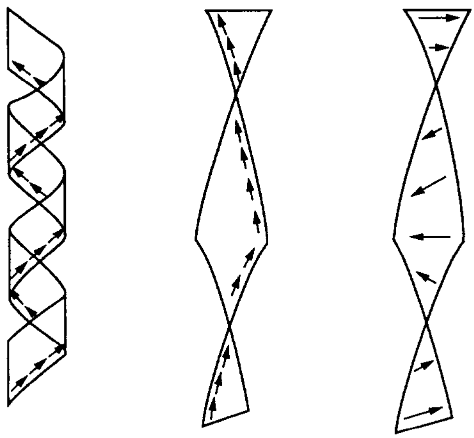


FIG. 3. Twist-folding of a paper strip into a helicoid shape.

dimensional maps and strange nonchaotic attractors on Sierpenski-like space. They may also be regarded as a two-dimensional section in an Anosov system which is shown to be typical for four-dimensional generalization of triadic Cantor sets.

III. The Discrete Peano-Hilbert Space—Fat Fractals

A peano-Hilbert space is an arcwise compact and locally connected metric space. It is thus essentially a space-filling curve as can be seen from the construction shown in Fig. 4(a) which is due to D. Hilbert.

In the following, the original peano curve (**8**) will serve as the basis of our discussion. For this form of the curve we need a lattice with at least nine squares (Fig. 4b). These squares define the 45°-rotated square shown in Fig. 4(c), whose Hausdorff dimension when iterated is easily shown in (**8**) to be 2. At the same time, the nine squares lie on two large peano blocks, as shown in Fig. 2. This curve, as is well known, possesses the properties of being self-avoiding and ergodic. Consequently, its capacity dimension will tend towards $d_c = 2$ and, as will be reasoned later, its information dimension will tend towards unity (**7**, **8**). As we repeat our procedure in the sense of a discrete iterated map, we see that each of the nine squares of all new sets of the small squares lies on smaller and smaller peano islands (Fig. 2), which we may now describe as discrete peano-Hilbert curves. More precisely, these curves have a double nature—they are globally discrete but locally connected, as can be seen from Fig. 2. The capacity dimension remains nevertheless the same. Superficially, this might be regarded as logically inconsistent at least in the classical geometrical sense, but again the notion of space-like curve is itself inconsistent in the same sense. In Fig. 2, we see how the nine squares lie in a larger vertical square mode of 25 squares. Finally, the entire dynamics of an S form

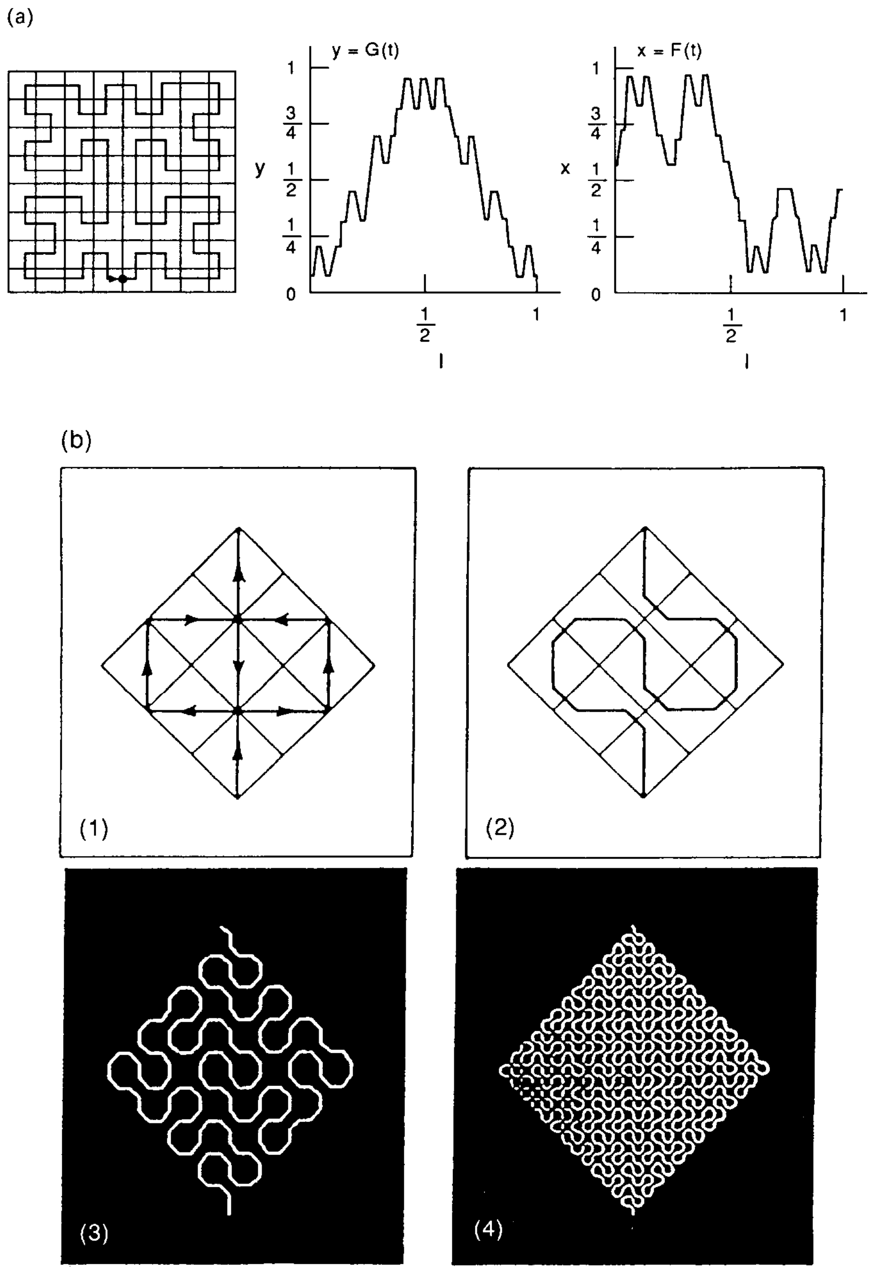
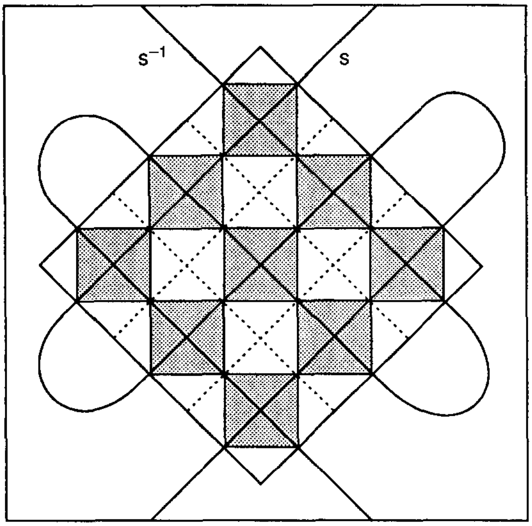


FIG. 4. The peano-Hilbert space filling curve ($d_c = \log 9/\log 3$). (a) Hilbert construction for E_2 peano space and the graph of the corresponding pair of parametric functions $x = F(t)$ and $y = G(t)$. (b) Construction of the original peano curve. (c) The S and S^{-1} intersections defining the invariant set for the first iteration of the peano map. (d) The three-dimensional peano-Hilbert curve. Micro space-time of quantum mechanics may well be similar to a four-dimensional analogue of this curve, as suggested in **(22, 24)**.

(c)



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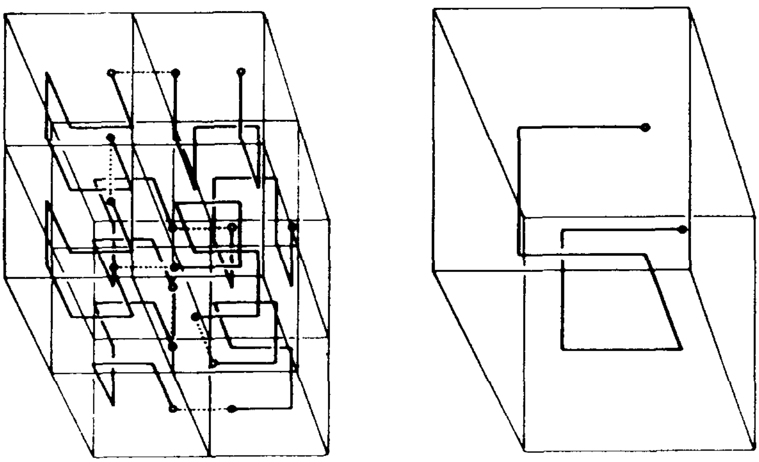

FIG. 4. *Continued.*

bending deformation of the pressed, stretched and twisted "phase space" lies in a seven by seven larger square. The invariant points of the set defining the starting point of the discrete peano curve construction now lie on the intersections of the S deformed shape indicated in Fig. 2, with arrows and their pre-image S^{-1}. A first estimation of the area left is $A \cong 0.6$ of the unit area. Note that the present notion of a discrete peano map is reminiscent of fate fractals where we have an area coverage of a finite portion of the total initial area but with holes in it on all scales. For a middle third Cantor set for instance, if we delete (1/3) for the nth iteration we obtain a fate fractal Cantor set with $d_c = 1$ instead of $d_c = 0.6309$ and a remaining length $l_F = 0.56$ of the original unit length. For a two-dimensional system with quasi-periodical orbits, Eubank and Farmer found a fat fractal set which also took $A_F \cong 0.56$ of the unit area of the full phase space (**11**). This is close to $A \cong 0.6$ found here.

IV. Relation to Quasi-periodic Forcing and Four-dimensional Phase Space

We have seen that a seemingly trivial change of symmetry due to a torsional deformation radically changes the capacity-Hausdorff dimension of the geometrical configuration of an idealized phase space. This movement mimics quasi-periodic forcing and the spiralling movement on a torus. In fact, our discrete map may be regarded in this sense as a quasi-periodically forced horseshoe-like map. This explains the numerically observed link between this form of forcing and the capacity dimension of possible strange attractors. Numerical calculations for these types of attractors have repeatedly shown for the Poincaré maps a capacity dimension near to $d_c = 2$, as well as a positive Lebesque measure (**1–4**). In the case of a pendulum for instance, one finds for both

$$\ddot{\phi}+0.005\dot{\phi}+0.027\sin\phi = 0.2\left[\cos t+\cos\frac{\sqrt{2}}{10}t\right], \quad (\dot{\ }) = \frac{\mathrm{d}(\ )}{\mathrm{d}t}$$

and

$$\ddot{\phi}+0.01\dot{\phi}+0.0272222\sin\phi = 0.15\left[\sin t+\sin\frac{\sqrt{2}}{10}t\right]\sin\phi$$

a strange nonchaotic attractor in the ϕ—$\dot{\phi}$ Poincaré map with a fractal capacity dimension

$$d_c \cong 1.81 \quad \text{and} \quad d_c \cong 1.904,$$

respectively. This suggests that the present model reflects the phenomenon of nonchaotic strange behaviour with reasonable accuracy.

One must, of course, remember that twist in a two-dimensional horseshoe is not permissible, which shows the crucial role played by the dimensionality of the phase space. In fact, for a four-dimensional phase space, such as that of a quasi-periodically forced pendulum, one could argue that "typically" a Poincaré map would have a fractal dimension $d_c = 2$ regardless of whether the attractor is chaotic or nonchaotic. This may be shown analytically by rescaling the phase dimension. This is discussed next.

V. Sierpenski Space and Quasi-ergodic Behaviour

The starting point of this analysis is Yorke's conjecture that single Cantor sets are the backbone of strange behaviour (**11**). To that we add the evident fact that

TABLE I

	$d_E^{(n)}$	$d_c^{(n)}$	c
Basic assumption	0	$d_c^0 = 0.63902$	$c = d_E^{(n)} - d_c^{(n)}$
Normality	1	1	
Results	2	1.58496	0.41504
	3	2.51210	0.48790
	4	3.98159	0.01841
	5	6.31067	−1.31067
	6	10.00218	−4.00218
	7	15.85309	−8.85309
	8	25.12655	−17.12655

the simplest fractal set is Cantor's triadic set (**12**) with $d_c = \log 2/\log 3$. If we accept these postulates then we can easily show that in four-dimensional phase space a strange set will typically have a Cantor-like fractal dimension $d_c \cong 4$. To arrive at this result, we must first find the equivalent of a triadic Cantor set in two dimensions. Such a set must be invariantly triadic Cantorian with regard to any linear or curvilinear transformation. Now we know that a unit area A of a Euclidian manifold is given by $A = (1)(1) = 1$. Consequently a corresponding quasi area of a Cantor set is $A_c = (d_c)(d_c)$. It follows then that in order to normalize A_c it must be multiplied by the normalization factor $\rho_2 = (A/A_c)_2$. Generalizing to n dimensions, we find $\rho_n = (V/V_c)_n$ where (V) stands now for hypervolume in n dimension. Denoting the nth Cantor-like fractal dimension in n-dimensional space by $d_c^{(n)}$ and the dimension of the corresponding Euclidian space in n dimensions by $d_E^{(n)} = n$, it follows then that†

$$d_c^{(n)} = \rho_n d_c = d_c/(d_c)^n = (1/d_c)^n = (d_s)^{n-1},$$

where d_s is termed the escalation factor. This is the totally symmetric set for which we are looking, and the result is now evaluated for $d_c = \log 2/\log 3$ in Table I, where we have introduced a new quantity termed *co-dimension* defined as $c = d_E^{(n)} - d_c^{(n)}$. Note that d_s could be equally interpreted as the Floquet multiplier of a discrete map $d_{(m+1)} = d_{(m)}/d_c^{(0)}$, where $n = m+1$.

There are a few interesting observations in Table I. First $d_c^{(n+1)}/d_c^{(n)} = d_s$ is the fractal dimension of a Serpenski space which is the prototype of fractal lattices with infinite hierarchy of semi loops and which is referred to frequently as the Serpenski gasket (**8**). Second, for all $n < 4$ we have $n > d_c^{(n)}$, while for $n > 4$ we have $d_c^{(n)} \gg n$. Only at $n = 4$ do we have a Cantor-like structure which comes very near to a space-filling set. The two-dimensional geometrical analogue of this is the peano-Hilbert curve which, as mentioned earlier, is ergodic and shares few properties with fat fractals (**11**, **13**). Therefore at $n = 4$ we share an almost ergodic set of the Anosov type. This point is clearly marked by the co-dimension c becoming

† It is stressed that the arguments used here to arrive at this formula cannot be regarded as mathematical derivation. Such a derivation was given, however, in (**22**).

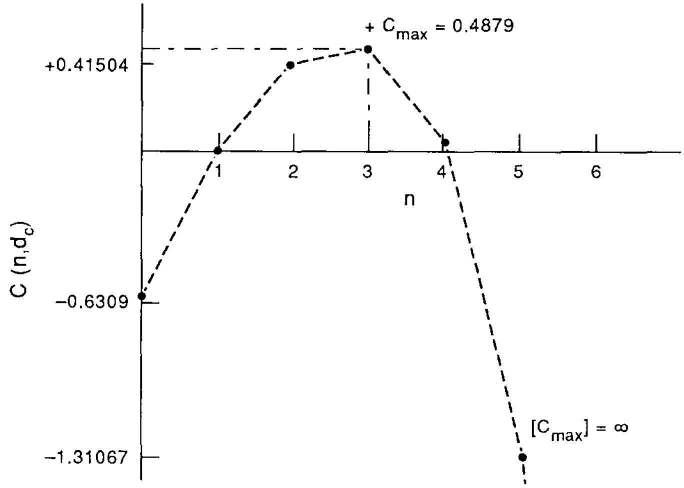

FIG. 5. The dependence of the co-dimension c on the dimensionality n of the Euclidian space E. For $c = 0$ we have topological saturation of E with fractals. The relationship is valid quantitatively only for $d_c^{(0)} = 0.6309$ and shows that in this case $c_{max} = 0.4879$ while $|c_{max}| = \infty$.

very small and then abruptly changing sign to negative as can be seen in Table I and Fig. 5.

It follows then that for a system with four-dimensional phase space ($n = 4$), a Poincaré section will typically have a capacity dimension

$$d_c \to \sim[(\sim 4)-(\sim 2)] \to \sim 2.$$

Here we have assumed that the map is very weakly dependent on the phase of forcing. The fact that the fractal dimension becomes substantially larger than the phase space dimension for $n = 4$ is a clear indication of a very rugged hypersurface of possible strange attractors and may be related to what was anticipated by Roessler. It also indicates that we will have self intersections and complete loops (Fig. 6) instead of the semi loops (Fig. 4b) and self-avoidance of the Sierpenski gasket and the peano-Hilbert space.

Finally, it is instructive to give an intuitive reasoning for the critical escalation value of $d_s = 1.58496$. Looking at Fig. 1(a), we see that this Cartesian product is triadic Cantorian along the perimetric and the diagonal directions but not in the two principal directions of the core as indicated. On the other hand, the Sierpenski carpet of Fig. 1(b) represents in effect the reverse situation where the principal directions of the core are triadic Cantor sets while the perimetric directions are totally connected. Consequently, an averaging of the capacity dimension of both sets must give a reasonable first approximation to a set with the symmetry properties of both sets. This expectation is confirmed by the very close proximity of $d_c = \frac{1}{2}[\log 4/\log 3 + \log 8/\log 3] = 1.577324$ and the capacity dimension of the Sierpenski gasket. The straightforward interpretation of $d_c^{(2)} = \log 3/\log 4$ is, of course, a scaling of $d_c^{(0)} = \log 2/\log 3$ proportional to the ratio of the area of the Euclidian

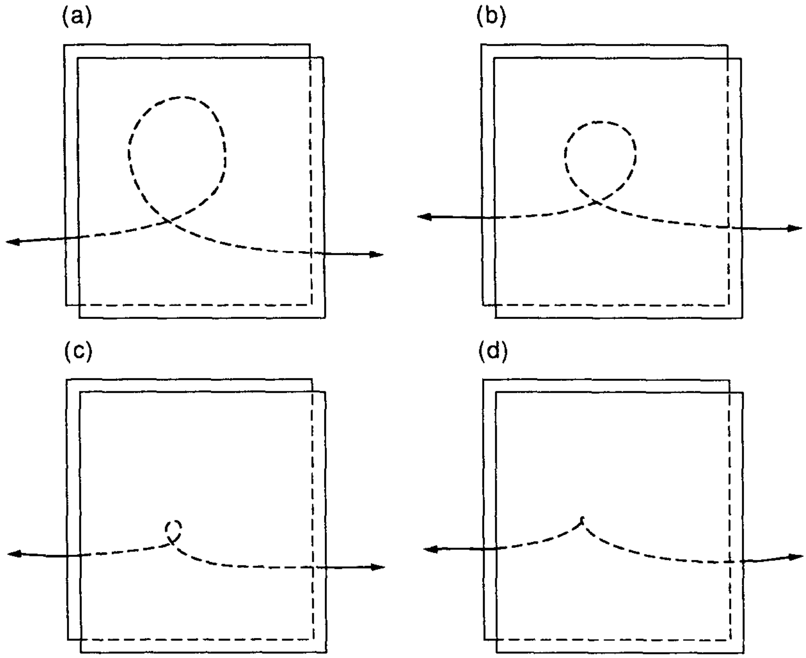

FIG. 6. The cusp as a limit of a homotopy of the loop.

phase space and the quasi area of a triadic Cantor space. Using similar averaging arguments the value $d_c^{(3)} = 2.51210$ of Table I could also be approximated by a Menger sponge, a three-dimensional Cartesian product of three triadic sets and a cube from which the core is removed. This gives

$$d_c = \tfrac{1}{3}[\log 21/\log 3 + \log 8/\log 3 + \log 26/\log 3] = 2.528423.$$

Again, this is very close to $d_c^{(3)} = 2.51210$.

VI. Insensitivity to Initial Conditions

Let us give first an intuitive explanation for the nonchaotic character of the peano-Hilbert dynamics using a trivial paper model. Taking a long strip of paper of length l and width a, where $l \gg a$, we start by drawing on it a chain of n squares so that $(l/n)/a = 1$. Now, fold only every second of the drawn squares in one direction along the diagonal. Subsequently, cut out all parts that overlap, except for a very thin seam to keep the chain connected. The result is a long chain identical to one row of the rotated peano squares of Fig. 2. We could have arrived at the same form by twisting the paper strip in many waves like a long helical spring (Fig. 3) then flattening it by simply pressing it on an even surface. The trivial, but important observation here is that the length of the strip is now reduced. The length reduction ratio is dependent on the ratio $(l/n)/a$ which is analogous to the frequency of quasiperiodic forcing. The shortening is clearly in the opposite direction to the axial stretching of the map. It follows then that analogous to one-dimensional horseshoe maps, the Liapunov exponents may be written as

$$\lambda_1 = \ln(\gamma - \beta), \quad \lambda_2 = -\ln \alpha\gamma,$$

where γ is the factor of stretching in one direction, α is the factor of compression in the perpendicular direction, while β is the factor of compression produced by the shortening due to torsional waves. Nonchaotic behaviour would consequently be associated with dynamics for which we have $0 \leqslant (\gamma - \beta) < 1$. It is important to observe here that the shortening factor which on its own is sufficient to become smaller than zero, corresponds to a rectangular deformation of the peano squares. This in turn causes the fractal dimension to drop under 2. For a rectangle with $a/b = 2$ for instance, we obtain $d = \log 32/\log 6 \cong 1.93$. We feel therefore that due to the inaccuracy and difficulty in calculating d_c, it is not easy to distinguish between chaotic and nonchaotic attractors based upon this criterion alone when d_c is very close to 2. However, we anticipate that in general, nonchaotic attractors will have $d_c \lesssim 2$ in addition to an information dimension $d_I \gtrsim 1$.

The nonchaoticity of our map could be argued in a different way. Ergodicity in all its different definitions always guarantees that the orbit covers the energy surface uniformly. It follows then that the pointwise dimension will be equal to its smooth Euclidian manifold, which in our case is two. We also know that the information dimension, which is not a simple metric-dimension, implies that the information entropy $H(\varepsilon)$ is an average of $N(1/\varepsilon)$ of the pointwise dimension. Combining these two facts for the "discrete" peano dynamic of our map it follows that $d_I \cong \frac{1}{2}d_p = 1$ for two dimensions. An intuitive explanation of this may be found by looking to the problem as being analogous to fat fractals. Using the Kaplan–Yorke conjecture about the equality of Liapunov dimension and the information dimension, it is a simple matter to show that $d_I = 1$ implies Liapunov exponent combination guaranteeing insensitivity to initial conditions, and thus non-chaoticity.

VII. Discussion

Four-dimensional dynamical systems are associated with wrinkled attractors and other interesting phenomena as pointed out in (**1–6**). We have contemplated the possible relevance of some of the numbers displayed in Table I in mathematical modelling. In particular, we found that $n = 4$ corresponds to the vanishing of the co-dimension c and topological saturation. In turn, this means that the dynamics is ergodic and eventually fills the entire phase space. Consequently, the Poincaré map is also area filling, which means $d_c \rightarrow 2$. This may provide an intuitive basis for the Ruelle, Takens and Newhouse theorem (**13**). According to Table I, the state for which $n = d_c^{(n)} = 4$ is a highly critical state because any further increase in dimension, say, $n = 5$, would obviously lead to self intersection replacing the semi loops of the Sierpenski construction by homoclinic loop soliton and even cusps as a homotopic limit for the loops (Fig. 6). This is so, because the fractal dimension $d_c^{(5)} \cong 6.3$ is significantly higher than $n = 5$ and consequently overlapping and intersection is inevitable.

A further interesting observation regarding Table I is that for any three successive dimensions $d_c^{(n)} \approx d_c^{(n-1)} + d_c^{(n-2)}$. This is a well-known Fibonacci property (**4**) and the corresponding dimension will be termed the Fibonacci fractal dimension (**14**). Should we insist that $d_c^{(n)} \approx d_c^{(n-1)} + d_c^{(n-2)}$ then we find that at $n = 4$ the corresponding Cantor-like dimension is $d_c = 4.23606$ while the Sierpenski space (**15**) is replaced by a "golden" space $d_s = 1/\Phi$, where Φ is the Golden mean (**12**). In fact, our Table I becomes identical to the table calculated by Cook (**16**) for Botticelli's Venus.

It might be interesting at this point to determine the escalation value d_s corresponding to the exact critical equality of $d_s^{(n)}$ and n in four dimensions. This is an elementary application of our formula relating $d_c^{(n)}$ to n. This way one finds

$$(1/d_c^{(n)})^{n-1} = (d)_s^{n-1} = n, \quad (1/d_c^{(4)})^{4-1} = d_s^{4-1}; \quad d_s = \sqrt[3]{4} \cong 1.587.$$

This is very close to the dimension of the Sierpenski space (**17**) $d_s = 1.58496$.

The role of multifractals in developing more refined mathematical models has not been discussed here.† Looking again at Table I, one may be led to reflect on fully developed turbulence and if at least a five-dimensional phase space is required to capture this phenomenon (**14**). This could be, for instance, a nonlinear set described by a phase space x, $\dot{x}$ and x' representing temporal and spatial oscillation of a behaviour space x. In addition, we need a spatial fluctuation w_x and a temporal fluctuation w_t as forcing frequencies. This makes them indeed five variables. The corresponding fractal dimension is thus $d_c^{(5)} \cong 6.3106$ which implies that we need at least seven dimensions for a reconstruction of the strange attractor of a fully developed turbulence from an experimentally obtained signal. Another interesting observation is that the Fibonacci fractal dimension $d_F^{(3)} = 1 + 1(\log 2/\log 3) = 2.584496$ is identical to $d_c = 1/(\log 2/\log 6)$, where $\log 2/\log 6$ is clearly a reasonable measure of the fractal dimension at period 3 chaos of a Feigenbaum cuscade. Note also that $d_c = \log 2/\log 6 = 0.387$ is very close to the smallest value ($d = 0.378$) found for period 3 chaos of the logistic map (**18**, **19**).

A point which is worth further careful investigation is the possible connection between loop soliton and chaos found recently in connection with diffusion-like process (**17**) and the Sierpenski structure which we hinted at earlier on.

On the other hand the connection between the loop and the cusp may easily be demonstrated using nothing more than a long fibre optic or guitar string. Deforming the string into a loop then applying a tensile force at the end points while forcing it to remain confined to two dimensions by holding it under a plate of glass, we see that the cusp is a homotopic limit of the loop, as shown in Fig. 6. Therefore, we conjecture that the phase space of two-frequency excited oscillators will eventually be covered completely with loops and cusps. This is nicely illustrated by the following application using the elastica-fluid particle path analogy (**17**).

The mathematical equation of the problem was shown in (**17**) to be nothing other than the well-known equation of the damped and excited pendulum or a

† It is not difficult to show that the sum of all $d_c^{(n)}$ from zero to infinity is $z = 2.709511$ for the triadic set and that $d_c^{(0)}$ is a kind of centre of gravity of the z set.

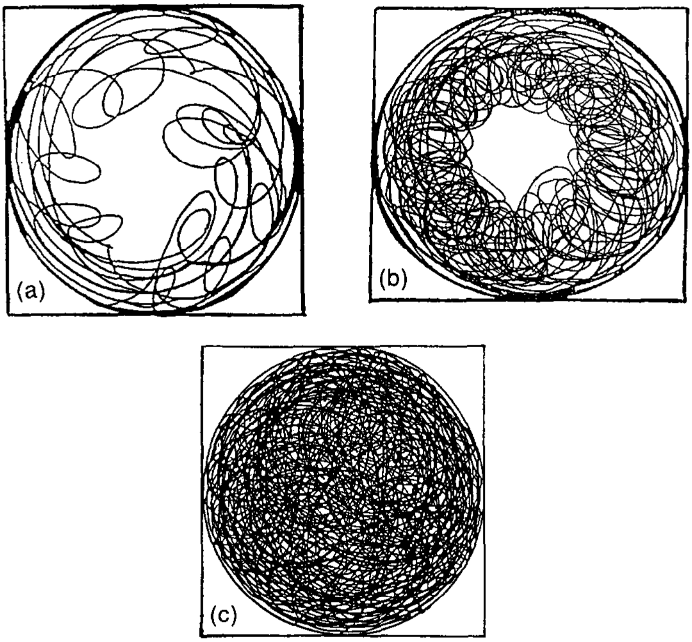

FIG. 7. Evolution of the chaotic path of a fluid particle in a circular container towards ergodicity. The particle is supposed to be pushed aside by a continuous steering using a cylindrical body. Using an elastica analogy (**17**), the figure may be interpreted as the chaotic loop soliton buckling of an elastic wire with axial periodic imperfection coiled infinitely many times inside a circular ridged cavity:

$$\phi''+0.15\phi'+\sin\phi = 0.94\sin 1.585\sin\phi, \quad (') = \mathrm{d}(\ )/\mathrm{d}x.$$

Note that the right-hand side of the above equation represents a spatial deterministic fluctuation which is referred to in engineering as shape imperfection (**19**).

Josephson junction. Now if we confine the imperfect elastica or the fluid to a circular ridged boundary, say a cylindrical tank, then this would be effectively an introduction of a second forcing frequency to the system making a quasi-periodically excited oscillator. In Fig. 7, we see how the system which could be interpreted now as the chaotic path of a fluid particle develops with the passing of time to an ergodic state. Similar behaviour may be found in the phugoids of gliding aircraft, billiards in magnetic fields, as well as many technically important problems associated with complex configurations of spatially chaotic systems (**19–21**). One such important problem is shown in Fig. 8.

VIII. Conclusion

Multidimensional dynamics based on perfect triadic Cantor sets leads to the conclusion that the fractal dimension of two frequencies quasi-periodically forced planar dynamical system is $d_c = 4$. Furthermore, it is conjectured that the maximum number of independent frequencies which may be observed, experimentally in a

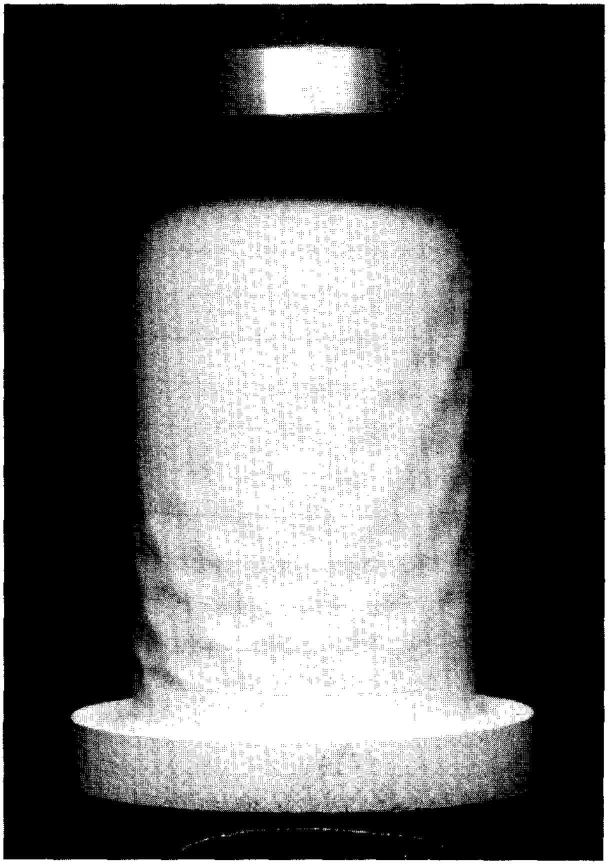

FIG. 8. Chaotic buckling configuration of a cylindrical shell under axial pressure. It is a form of what may be called elastostatical turbulence (**17**). In this particular experiment, due to Walker and El Naschie (**19**), a curious quasi-symmetric pattern is the final configuration as noted in (**23**).

quasi-periodic oscillation with any likelihood is four. This would correspond to a five-dimensional system with a theoretical Cantor-like fractal dimension $d_c^{(5)} \cong 6.31$ and may be relevant to the study of fully developed (strong) turbulence. It is interesting to note that all these conclusions are model-independent. This means that they do not depend on the precise details of any particular system. They depend mainly on the properties of Cantor-like sets, the topology and dimensionality of the involved spaces and some number theoretical properties. Finally, one may remark that four-dimensional Cantor sets may be relevant to modelling the geometry of micro space-time, as suggested in (**22**).

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